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A three-dimensional laminar boundary-layer calculation

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Results are presented of a preliminary numerical investigation into a three-dimensional laminar boundary layer. It is assumed that the flow is over a developable surface and the boundary conditions at the outer edge of the layer are chosen to be $u = U_0 + xU_1$, $v = yV_1$. This choice enables the governing equations to be written in terms of two, and not three, independent variables, viz. x and z. However, the three-dimensionality of the problem gives rise to a coupling of the equations which, not unnaturally, is still present after the elimination of y.

For appropriate values of U_1 and V_1 it is found possible to integrate the equations approximately from the 'birth' of the boundary layer ($x = 0$) right up to a saddle point of attachment. Calculations have already been made for flow at such attachment points and the comparison of the present results with them is extremely good.

1. Introduction

When a fluid flows past a finite body there are points or lines at which the fluid (i) attaches itself to the body, and (ii) separates from the body. For example, in the two-dimensional flow past a circular cylinder placed perpendicularly (or yawed) to the on-coming stream, the fluid attaches itself to the front generator and leaves at some other generators, while for a sphere the attachment occurs at a point.

If, at the surface of a body, we write $\boldsymbol{\omega}_0$ for the fluid vorticity and $\mu\boldsymbol{\epsilon}_0$ for the skin friction, then the differential equations for the skin friction and vortex lines are

$$\mathbf{dr} = K_1\boldsymbol{\epsilon}_0, \quad \mathbf{dr} = K_2\boldsymbol{\omega}_0$$

respectively. A point at which $\boldsymbol{\epsilon}_0$ and $\boldsymbol{\omega}_0$ vanish is called a stagnation point and is a singular point of these differential equations. The classification of such points depends on the sign of the Jacobian, $J = \partial(\epsilon_x, \epsilon_y)/\partial(x, y)$ where (x, y, z) is a co-ordinate system with the singular point as origin. If $J > 0$ the point is a nodal point and if $J < 0$ it is a saddle point.

The normal velocity w (i.e. in the direction of z) near the singular point is given by

$$w = -\tfrac{1}{2}\Delta z^2 + O(z^3),$$

where $\Delta = \boldsymbol{\nabla}.\boldsymbol{\epsilon}_0$ is the two-dimensional divergence of $\boldsymbol{\epsilon}_0$. So that depending on whether $\Delta > 0$ or < 0, the singular point is a stagnation point of attachment or separation respectively.

The flow at nodal points of attachment has been discussed by Howarth (1951), who shows that by choosing the orthogonal axes in a suitable manner the curvature terms in the equations can be neglected. The problem was reduced to the solution of two coupled, third-order, non-linear differential equations involving a parameter c which is a measure of the ratio of the two curvatures at the stagnation point.

For nodal points of attachment $c > 0$ and it is sufficient to consider the range $0 \leqslant c \leqslant 1$. Howarth (1951) gives solutions for $c = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The case when $c < 0$ corresponds to saddle points which are, if $0 > c > -1$, also points of attachment. Davey (1961) has examined the equations for $c < 0$ and obtained solutions for $0 > c \geqslant -1$, and also shown that for $c < -1$, which corresponds to saddle points of separation, the equations cannot then be solved. We may also note that for $-1 \leqslant c < -0{\cdot}4294$ 'flow reversal' occurred, although it was still possible to get numerical solutions.

Figure 1 shows the sort of situation where these solutions may be valid. The point N corresponds to a nodal point of attachment where Howarth's solution is probably operative. Varying c (i.e. the curvatures at N) gives results which seem

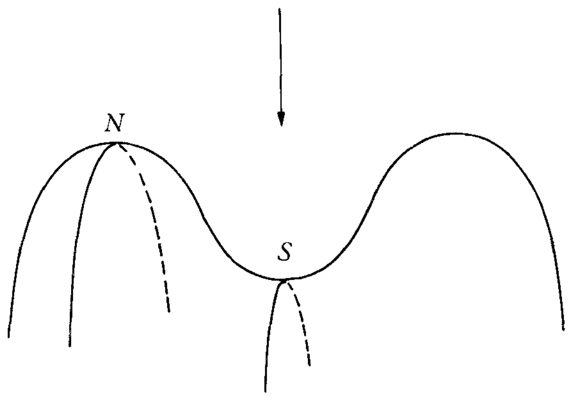

FIGURE 1. Typical nodal (N) and saddle (S) points. The arrow indicates the flow direction.

to agree with an intuitive picture that one may reasonably form. The point S corresponds to a saddle point of attachment and if Davey's results are at all relevant it is to this region that they apply. Here again his results seem to agree with intuitive ideas.

However, the assumption which underlies Davey's work is that the flow at such saddle points of attachment is locally determinate. There are other flows which have been found to be locally determinate (e.g. the flow near the centre of a finite disk in rotating fluid, Rogers & Lance (1964)) but it is not at all clear what conditions are to be satisfied to make them so. An argument has been given for the present problem (Rosenhead 1963, p. 78) which says, in effect, that it is determinate locally since the fluid has been under the influence of the region S long enough to make it so.

Although the present results confirm this, it was thought to be non-trivial enough to make worth while a verification by tracing a boundary layer from its birth to a saddle point of attachment, and this was the reason for these calculations.

2. The boundary-layer problem

In Cartesian co-ordinates the boundary-layer equations governing the flow over a three-dimensional developable surface are

$$\left.\begin{aligned} u'\frac{\partial u'}{\partial x}+v'\frac{\partial u'}{\partial y}+w'\frac{\partial u'}{\partial z} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x}+\nu\frac{\partial^2 u'}{\partial z^2},\\ u'\frac{\partial v'}{\partial x}+v'\frac{\partial v'}{\partial y}+w'\frac{\partial v'}{\partial z} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y}+\nu\frac{\partial^2 v'}{\partial z^2},\\ \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z} &= 0,\end{aligned}\right\} \tag{1}$$

where (u',v',w') are the velocity components in the directions (x,y,z) respectively. The problem to be investigated herein is the solution of these equations subject to the boundary conditions

$$u' = v' = w' = 0 \quad \text{on} \quad z = 0,$$
$$u' \to U_0 + xU_1, \quad v' \to yV_1 \quad \text{as} \quad z \to \infty,$$

where U_0, U_1, V_1 are constants with $U_0 > 0$.

For $V_1 \equiv 0$ the problem reduces to the two-dimensional flow past a semi-infinite flat plate with pressure gradient (providing $U_1 \neq 0$). With V_1 non-zero the flow can be interpreted as fluid moving over a suitably curved plate, or, equivalently, moving over a flat plate above which a suitably shaped body has been placed. In all these cases the boundary layer's 'birth' occurs at the leading edge and develops downstream in the x-direction.

The reason for choosing these particular boundary conditions was, first, to try to reach a downstream stagnation point with no separation and, secondly, to simplify the equations in (1). The simplification is achieved by noting that a solution of these equations can be obtained by writing $u' = u(x,z)$, $v' = yv(x,z)$, $w' = w(x,z)$. The equations for u, v, w are

$$\left.\begin{aligned} u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2},\\ u\frac{\partial v}{\partial x}+v^2+w\frac{\partial v}{\partial z} &= V_1^2+\nu\frac{\partial^2 v}{\partial z^2},\\ \frac{\partial u}{\partial x}+v+\frac{\partial w}{\partial z} &= 0,\end{aligned}\right\} \tag{2}$$

and the boundary conditions are

$$\begin{gathered} u = v = w = 0 \quad \text{on} \quad z = 0,\\ u \to U_0 + xU_1, \quad v \to V_1 \quad \text{as} \quad z \to \infty. \end{gathered} \tag{3}$$

Although total elimination of the independent variable y has been achieved, which results in a far more tractable problem, the equations remain coupled.

In the following we are concerned with the solution of equations (2), subject to (3), for various values of the constants U_1 and V_1. However, before going into detail it will be convenient to speculate on the probable flow characteristics.

With $V_1 \equiv 0$ and $U_1 < 0$ we have Howarth's (1938) two-dimensional flow problem with separation† at $U_1 x/U_0 = -0{\cdot}120$. With V_1 small but positive it seems plausible that the effect is to bleed away the retarded boundary-layer fluid and hence delay separation. Wilkinson (1954) has considered a perturbation approach using the Pohlhausen method and showed that these ideas are probably correct. Similar reasoning when V_1 is negative suggests that the boundary layer is thickened by feeding in fluid leading to earlier separation. This is also in accord with Wilkinson's results.‡

If V_1 is increased beyond a certain multiple of $-U_1$ it is possible that separation will be completely eliminated (in contrast to the two-dimensional problem with suction) and thus the stagnation point at $x = -U_0/U_1$ will be within the range of the calculation.

The above is only part of the flow spectrum, but it is enough to indicate the necessity of approaching the problem in two ways: the first is a series approach, modified by one of Howarth's techniques, to deal with the separating flows, and the second is a Pohlhausen calculation to give approximately the overall picture but which we may anticipate as being fairly reliable for the non-separating flows.

Full details regarding the methods and results are given in the account presented in Banks (1963), although the notation differs slightly.

All computations were done on the Mercury computers at Oxford and London Computing Centres using the Autocode system.

3. Solution by series

We introduce a three-dimensional vector potential such that

$$u = \frac{\partial \psi}{\partial z}, \quad v = \frac{\partial \phi}{\partial z}, \quad w = -\phi - \frac{\partial \psi}{\partial x},$$

where $\psi = \psi(x,z)$ and $\phi = \phi(x,z)$. For convenience in comparing certain of our results with those of Howarth (1938) we write

$$\psi = \left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}} \{f_0 + (8x) f_1 + (8x)^2 f_2 + \ldots\},$$

$$\phi = \left(\frac{\nu x}{U_0}\right)^{\frac{1}{2}} \{g_0 + (8x) g_1 + (8x)^2 g_2 + \ldots\},$$

where f_i and g_i are functions of $Z = \frac{1}{2} z (U_0/\nu x)^{\frac{1}{2}}$.

The continuity equation in (2) is automatically satisfied by this choice of ψ and ϕ. The first momentum equation gives

$$\sum_{r=0}^{n} 2r f'_r f'_{n-r} - \sum_{r=0}^{n} (2r+1) f_r f''_{n-r} - \frac{1}{4} \sum_{r=1}^{n} g_{r-1} f''_{n-r} = U_0 U_1 \delta_{n1} + \tfrac{1}{8} U_1^2 \delta_{n2} + U_0 f'''_n,$$

† Separation here, and henceforth, will be used to denote the vanishing of $\partial u'/\partial z$ and $\partial v'/\partial z$ at $z = 0$.

‡ Ahuja (1964), using Wilkinson's method, investigates the effect of suction and injection on such a boundary layer.

with boundary conditions

$$f_n(0) = f'_n(0) = 0,$$

$$f'_n(\infty) = 2U_0\delta_{n0} + \tfrac{1}{4}U_1\delta_{n1},$$

where

$$\delta_{ij} = 1 \quad \text{if } i = j,$$
$$= 0 \quad \text{otherwise,}$$

and dashes imply differentiation with respect to Z. The second momentum equation gives

$$\sum_{r=0}^{n} 2rg'_r f'_{n-r} + \frac{1}{4}\sum_{r=1}^{n}(g'_{r-1}g_{n-r} - g''_{n-r}g_{r-1}) - \sum_{r=0}^{n}(2r+1)g''_{n-r}f_r = V_1^2\delta_{n1} + U_0 g'''_n,$$

with boundary conditions

$$g_n(0) = g'_n(0) = 0,$$

$$g'_n(\infty) = 2V_1\delta_{n0}.$$

If we write $f_i = U_0(U_1/U_0)^i F_i$ and $g_i = V_1(U_1/U_0)^i G_i$ it is found that $G_0 = F_0$ where

$$F'''_0 + F_0 F''_0 = 0$$

and

$$F_0(0) = F'_0(0) = 0; \quad F'_0(\infty) = 2.$$

Also with

$$P_n \equiv \frac{d^3}{dZ^3} + F_0\frac{d^2}{dZ^2} - 2nF'_0\frac{d}{dZ} + (2n+1)F''_0,$$

$$Q_n \equiv \frac{d^3}{dZ^3} + F_0\frac{d^2}{dZ^2} - 2nF'_0\frac{d}{dZ},$$

the equations for F_n and G_n ($n \geqslant 1$) are

$$P_n(F_n) = \sum_{r=1}^{n-1}\{2rF'_{n-r}F'_r - (2r+1)F''_{n-r}F_r - \tfrac{1}{4}\alpha F''_{n-r}G_{r-1}\} - \tfrac{1}{4}\alpha F''_0 G_{n-1} - \delta_{n1} - \tfrac{1}{8}\delta_{n2}, \tag{4}$$

$$Q_n(G_n) = \sum_{r=1}^{n-1}\{2rF'_{n-r}G'_r + \tfrac{1}{4}\alpha(G'_{n-r}G'_{r-1} - G''_{n-r}G_{r-1})\} - \sum_{r=1}^{n}(2r+1)G''_{n-r}F_r + \tfrac{1}{4}\alpha(G'_0G'_{n-1} - G''_0G_{n-1}) - \alpha\delta_{n1}, \tag{5}$$

where $\alpha = V_1/U_1$. The boundary conditions to be satisfied are

$$F_n(0) = F'_n(0) = 0, \quad F'_n(\infty) = \tfrac{1}{4} \quad \text{if} \quad n = 1,$$
$$= 0 \quad \text{otherwise,}$$

$$G_n(0) = G'_n(0) = 0, \quad G'_n(\infty) = 0.$$

These equations contain terms involving α, although, if we use Howarth's (1935) method of splitting, it is possible to write F_n, G_n ($n > 0$) in terms of universal functions whose governing differential equations and boundary conditions are independent of α. To do this we write

$$F_0 = F_{01}$$

and for $n > 0$

$$F_n = \sum_{r=1}^{n+1}\alpha^{r-1}F_{nr}, \tag{6}$$

$$G_n = \sum_{r=1}^{n+1}\alpha^{r-1}G_{nr}. \tag{7}$$

The expressions for u and v become

$$u = \tfrac{1}{2}U_0\{F'_{01} + (8X)B(F'_{11} + \alpha F'_{12}) + (8X)^2(F'_{21} + \alpha F'_{22} + \alpha^2 F'_{23}) + (8X)^3 B(F'_{31} + \alpha F'_{32} + \alpha^2 F'_{33} + \alpha^3 F'_{34}) + \ldots\},$$

$$v = \tfrac{1}{2}V_1\{F'_{01} + (8X)B(G'_{11} + \alpha G'_{12}) + (8X)^2(G'_{21} + \alpha G'_{22} + \alpha^2 G'_{23}) + (8X)^3 B(G'_{31} + \alpha G'_{32} + \alpha^2 G'_{33} + \alpha^3 G'_{34}) + \ldots\},$$

where $X = |U_1|x/U_0$ and $B = U_1/|U_1|$. Full details of this part of the investigation are given in Banks (1963) and will not be reproduced here.

With $\alpha = 0$ (i.e. $V_1 = 0$) the calculation reduces to that of Howarth (1938), although for $\alpha \neq 0$ the number and complexity of the differential equations is much greater. A measure of the complexity of the three-dimensional problem over that of the two-dimensional one is illustrated by noting that, as given by Howarth, the number of terms on the right-hand side of the differential equation defining f_8 is 11, whereas in the present problem there are 10 terms on the right-hand side of the equation defining F_{33}. In fact the problem, even to terms of $O(X^4)$, was too big for Mercury using the Autocode system and the solution of G_{43} and G_{44} could not be obtained in this way. This difficulty was overcome by solving (5) (with $n = 4$) for two non-zero values of α and calculating $G''_{43}(0)$, $G''_{44}(0)$ using (7).†

We note here that for α infinitely large the boundary-layer solution becomes

$$\left.\begin{aligned} u &= \tfrac{1}{2}U_0\{F'_{01} + (8\bar{X})B_1 F'_{12} + (8\bar{X})^2 F'_{23} + (8\bar{X})^3 B_1 F'_{34} + \ldots\}, \\ v &= \tfrac{1}{2}V_1\{F'_{01} + (8\bar{X})B_1 G'_{12} + (8\bar{X})^2 G'_{23} + (8\bar{X})^3 B_1 G'_{34} + \ldots\}, \end{aligned}\right\} \tag{8}$$

where $\bar{X} = x|V_1|/U_0$ and $B_1 = V_1/|V_1|$. This corresponds to the external flow $U = U_0$, $V = yV_1$ of course, and any comparison with results for $|\alpha|$ large but finite must be made using the independent variable $\bar{X} = |\alpha|X$.

Now in the case of the two-dimensional linearly retarded flow Howarth (1938) estimated that at least 15 terms of the series expansion were required in order to calculate the skin friction near separation. The reason for this appears to be due to a singularity in the boundary-layer equations at separation. Although the present problem is essentially a three-dimensional one, it seems reasonable to expect that some sort of singularity will arise. Terrill (1960) has shown that for the two-dimensional boundary layer with suction the singularity is still present at separation.

For the case of a separating boundary layer, the skin friction, and hence the position of separation, can be found reasonably accurately by using one of the methods suggested by Howarth (1938), whereby the contribution from the terms up to $O(X^m)$ is regarded as an approximate solution and then u, v are written

$$\left.\begin{aligned} u &= \tfrac{1}{2}U_0\{F'_0 + (8X)BF'_1 + (8X)^2 F'_2 + \ldots + (8BX)^{m-1}F'_{m-1} + A(X)F'_m\}, \\ v &= \tfrac{1}{2}V_1\{F'_0 + (8X)BG'_1 + (8X)^2 G'_2 + \ldots + (8BX)^{m-1}G'_{m-1} + C(X)G'_m\}. \end{aligned}\right\} \tag{9}$$

† Since this work was completed a more general series calculation has been made by Sowerby (1965), who investigates the three-dimensional effects near the leading edge of a flat plate. He considers terms to $O(X^2)$ only and uses different variables. However, comparison of the appropriate terms provides a check to $O(X^2)$.

The correction terms are governed by the functions $A(X)$ and $C(X)$ and these are determined by using the two compatibility relations obtained by differentiating the boundary-layer equations twice with respect to z and then evaluating at $z = 0$. The two conditions to be satisfied are

$$\left.\begin{aligned} \left(\frac{\partial u}{\partial z}\right)_0 \left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)_0 - \left(\frac{\partial v}{\partial z}\right)_0\right\} &= \nu\left(\frac{\partial^4 u}{\partial z^4}\right)_0 \\ \text{and} \qquad 2\left(\frac{\partial u}{\partial z}\right)_0 \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)_0 - \left(\frac{\partial v}{\partial z}\right)_0 \left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right)_0 - \left(\frac{\partial v}{\partial z}\right)_0\right\} &= \nu\left(\frac{\partial^4 v}{\partial z^4}\right)_0 . \end{aligned}\right\} \tag{10}$$

The prime interest here is not so much in the functions $A(X)$ and $C(X)$ as such, but in the contribution they give to the skin friction. We therefore substitute the differentiated expressions of (9) in (10) and then, as Curle (1960) did in a two-dimensional investigation, eliminate the functions $A(X)$ and $C(X)$ by using (9) again, so that we have two equations which determine the two components of the skin friction.

Of course, the assumptions implicit in writing equations (9) in this form are that the extra terms $A(X)F'_m$ and $C(X)G'_m$ are small and that the functions F_p and G_p for $p \geqslant m+1$ are respectively similar in shape to the functions F_m and G_m. The smallness of the added terms can be justified at a later stage although the similarity of the functions F_p and G_p is only shown to be reasonable by examining F_{m-2}, F_{m-1}, F_m and G_{m-2}, G_{m-1}, G_m.

Writing
$$L = \frac{4}{U_0}\left(\frac{\nu X}{|U_1|}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial z}\right)_0, \quad M = \frac{4}{V_1}\left(\frac{\nu X}{|U_1|}\right)^{\frac{1}{2}}\left(\frac{\partial v}{\partial z}\right)_0,$$

the equations for L and M are

$$\left.\begin{aligned} XL\frac{dL}{dX} - \frac{L^2}{2} - B\alpha XLM &= \frac{1}{2}\left\{H_0 + \ldots + (8BX)^{m-1}H_{m-1} + L\frac{F^{\mathrm{v}}_m(0)}{F''_m(0)}\right\} \\ \text{and} \qquad & \\ 2XL\frac{dM}{dX} - \frac{LM}{2} + B\alpha XM^2 - XM\frac{dL}{dX} & \\ &= \frac{1}{2}\left\{E_0 + \ldots + (8BX)^{m-1}E_{m-1} + M\frac{G^{\mathrm{v}}_m(0)}{G''_m(0)}\right\}, \\ \text{where} \qquad & \\ H_i = F^{\mathrm{v}}_i(0) - F''_i(0)\,[F^{\mathrm{v}}_m(0)/F''_m(0)], \quad E_i &= G^{\mathrm{v}}_i(0) - G''_i(0)\,[G^{\mathrm{v}}_m(0)/G''_m(0)] \end{aligned}\right\} \tag{11}$$

and superscripts denote differentiation with respect to Z. The initial values for L and M so as to start the integration of equations (11) can be obtained from the following relations:

$$L = F''_0(0) + (8X)BF''_1(0) + \ldots,$$
$$M = F''_0(0) + (8X)BG''_1(0) + \ldots.$$

The functions $F''_i(0)$, $G''_i(0)$, etc., and hence L and M, are all functions of α and the equations in (11) have been integrated for a number of values of α which are consistent with the assumptions made.

The functions $F^{\mathrm{v}}_i(0)$, $G^{\mathrm{v}}_i(0)$ are obtained by differentiation of the appropriate differential equations and it is found that

$$F^{\mathrm{v}}_0(0) = -[F''_0(0)]^2,$$

while, for $n \geqslant 1$,

$$F_n^{\mathrm{v}}(0) = \sum_{r=1}^{n} (2r-1) F_r'''(0) F_{n-r}''(0) - \frac{\alpha}{4} \sum_{r=1}^{n} G_{r-1}'''(0) F_{n-r}''(0),$$

$$G_n^{\mathrm{v}}(0) = \sum_{r=0}^{n-1} (4n-6r-1) F_r'''(0) G_{n-r}''(0) + \frac{\alpha}{4} \sum_{r=1}^{n} G_{r-1}'''(0) G_{n-r}''(0) - (2n+1) F_0''(0) F_n''(0).$$

Results of series method

Using the universal function approach it was found that, with the exception of G_{43} and G_{44}, all the functions up to $O(X^4)$ could be integrated numerically. $G_{43}''(0)$ and $G_{44}''(0)$ were obtained using (5), and table 1 gives $F_{ij}''(0)$ and $G_{ij}''(0)$. Tabulation of the universal functions is given in Banks (1963), and good agreement was found on comparison of the appropriate results with those obtained by Howarth (1938).

$F_{ij}''(0)$

$i \backslash j$	1	2	3	4	5
0	1·328229	—	—	—	—
1	1·020541	0·083014	—	—	—
2	−0·069253	−0·025473	0·012189	—	—
3	0·055960	0·019079	−0·005206	−0·001017	—
4	−0·03718	−0·01586	0·00285	0·00101	−0·00004

$G_{ij}''(0)$

$i \backslash j$	1	2	3	4	5
1	0·311526	0·626000	—	—	—
2	−0·083250	−0·094731	−0·021748	—	—
3	0·04766	0·04129	0·00676	−0·00194	—
4	−0·03290	−0·02673	−0·00354	0·00185	0·00038

TABLE 1

Similarity of the functions has been examined (Banks 1963) by plotting $F_i'(Z)/F_i(\infty)$ and $G_i'(Z)/G_i(\infty)$ ($i = 2, 3, 4$) for various values of α. It appears that for $|\alpha|$ not very large ($-2 < \alpha < 2$) similarity exists, but as $|\alpha|$ increases from zero it gets progressively worse. This failure to deal with the whole range of α is not unexpected. Nevertheless, we may anticipate that, although Howarth's method is not applicable for a great range of α values, it is highly likely that the Pohlhausen results will, to a certain extent, be able to cope over this range.

The equations in (11) for the x- and y-components of the skin friction (i.e. $L(X, \alpha)$ and $M(X, \alpha)$ respectively) were integrated for a number of values of α. The results are given graphically in figures 2 and 3 for $B = -1$. The integration of these equations was also attempted for larger values of $|\alpha|$ (i.e. $\alpha < -2{\cdot}5$ and $\alpha > 3{\cdot}5$) with $B = -1$ but the resulting values of $A(X, \alpha)$ and $C(X, \alpha)$ were fairly large for moderate values of X, and this was taken to imply breakdown of the underlying assumptions necessary for the use of this procedure. The bounds on α found in this way are certainly consistent with those suggested by the similarity requirements, but naturally no absolute bounds can be found in this

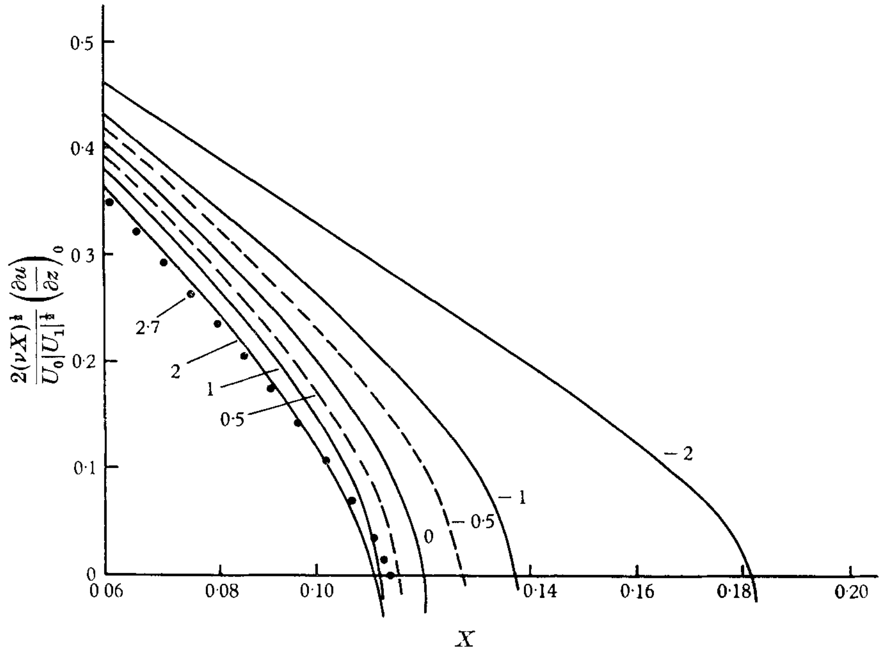

FIGURE 2. The variation of the x-component of skin friction from the modified-series results. The values of α are indicated on the figure.

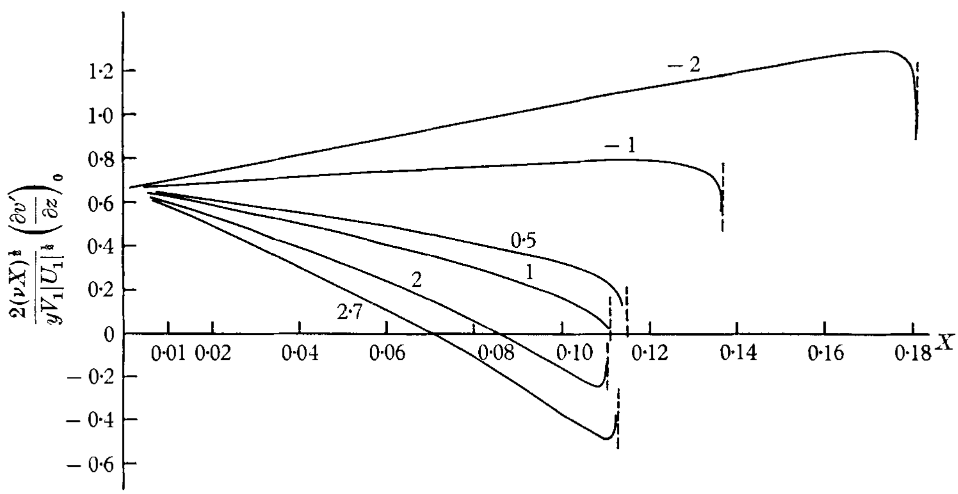

FIGURE 3. The variation of the y-component of skin friction from the modified-series results for the values of α shown.

manner. Results for $B = +1$ are given in Banks (1963) and we merely note here that even with $U_1 = 0$ and $V_1 < 0$ no separation appears possible.

With $\alpha = 0$ ($U_1 < 0$) we find separation is predicted at $X = 0{\cdot}1194$, which should be compared with the values of $0{\cdot}120$ and $0{\cdot}1198$ as given by Howarth (1938) and Leigh (1955) respectively. Hence reasonable agreement is obtained in this special case, and there appears to be no immediate reason why the values of separation presented here for $\alpha \neq 0$ should not have a similar accuracy.

These findings appear to confirm some of the plausible arguments given earlier, in the sense that for $|\alpha| < 1$ separation of the linearly retarded flow is delayed or advanced according as $V_1 > 0$ or $V_1 < 0$. In fact the results seem to suggest (and it is confirmed by the approximate solution given in §4) that for $|\alpha|$ large enough separation is eliminated altogether. This latter phenomenon is not totally unexpected for the case $V_1 > 0$ (i.e. $\alpha < 0$) but the fact that there is a strict minimum for $X_s(\alpha)$ at about $\alpha = 2$ and that, furthermore, for $\alpha \geqslant 3{\cdot}5$ boundary-layer separation is again inhibited, appears to be a novel one.

Further discussion of these results, together with a comparison with the Pohlhausen analysis of §4, is given in §5.

	$\frac{2\nu^{\frac{1}{2}}X^{\frac{1}{2}}}{U_0\|U_1\|^{\frac{1}{2}}}\left(\frac{\partial u}{\partial z}\right)_0$		$A(X)$	
X	Howarth	Present results	Howarth	Present results
0·0625	0·395	0·394	0·094	0·09
0·0750	0·336	0·335	0·21	0·24
0·0875	0·271	0·270	0·47	0·53
0·1000	0·199	0·198	0·93	1·15
0·1125	0·109	0·106	2·12	2·67
0·1150	—	—	2·68	3·35
0·1194	—	0·000	—	—
0·120	0·000	—	—	—

TABLE 2

Before proceeding to the approximate method it is of interest to make a more detailed comparison with Howarth's (1938) results. There are slight differences in procedure between his approach and that used here and, furthermore, he had obtained more terms in the series expansion, so that complete agreement is unlikely. However, they should be reasonable enough to provide a partial check on the above analysis and integration.

In Howarth's treatment he finds that (in his notation) f_5', f_6', f_7', and f_8' can be fairly accurately expressed in the form $K_r \eta\, e^{-0\cdot 1\eta^3}$ where K_r is a constant depending on f_r, and η is equivalent to Z which has been defined above. He then writes

$$u = \tfrac{1}{2}b_0\{f_0' - (8x^*)f_1' + \ldots + (8x^*)^6 f_6' + \phi(x^*)\,\eta\, e^{-0\cdot 1\eta^3}\},$$

where b_0 and x^* are respectively U_0 and X in the present notation. Also f_0, f_1, f_2, etc., are the same functions as F_{01}, F_{11}, F_{21}, etc., of the present work.

Hence

$$\frac{4\nu^{\frac{1}{2}}x^{\frac{1}{2}}}{b_1^{\frac{1}{2}}b_0}\left(\frac{\partial u}{\partial z}\right)_0 = f_0''(0) - (8x^*)\,f_1''(0) + \ldots + (8x^*)^6 f_6''(0) + \phi(x^*),$$

and comparison with the above indicates that

$$A(X)\,F_{41}''(0) = (8x^*)^4 f_4''(0) - (8x^*)^5 f_5''(0) + (8x^*)^6 f_6''(0) + \phi(x^*)$$

or, since $F_{41}''(0) = f_4''(0)$,

$$A(X) = (8X)^4\left\{1 - (8X)\frac{f_5''(0)}{f_4''(0)} + (8X)^2\frac{f_6''(0)}{f_4''(0)}\right\} + \frac{\phi(X)}{f_4''(0)},$$

so that a direct comparison with Howarth's results is possible. Table 2 gives both sets of results for the skin friction and the function $A(X)$ for various X.

As will be noted, agreement is very reasonable apart from the values of $A(X)$. Fortunately the latter does not appear to be critical with regard to the skin friction and separation.

4. Pohlhausen solution

The main reason for the calculation in this section was the desire to obtain a fairly quick and general guide as to the overall behaviour of the flow pattern; a measure of the success of the Pohlhausen method is given in §5. Furthermore, since the exact series solution was also being currently considered, it was thought desirable to compare the one boundary-layer thickness method of Taylor (1950) with the two boundary-layer thickness approach of Cooke (1952). However, it was found that both methods gave virtually identical results and so only the one boundary-layer thickness method and the results are presented here. The reader is again referred to Banks (1963) for further details.

Using equations (2) and the boundary conditions (3) the momentum integral equations are found to be

$$\left.\begin{aligned} \frac{d}{dx}\{U^2(\theta_1+\delta_1)\} - U\frac{d}{dx}(U\delta_1) + V_1\gamma_1 &= \nu\left(\frac{\partial u}{\partial z}\right)_0, \\ \frac{d}{dx}(UV_1\gamma_2) + V_1^2(\theta_2+\delta_2) + V_1^2\theta_2 &= \nu\left(\frac{\partial v}{\partial z}\right)_0, \end{aligned}\right\} \tag{12}$$

where

$$\delta_1 = \int_0^\infty \left(1-\frac{u}{U}\right)dz, \qquad \delta_2 = \int_0^\infty \left(1-\frac{v}{V_1}\right)dz,$$

$$\theta_1 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dz, \quad \theta_2 = \int_0^\infty \frac{v}{V_1}\left(1-\frac{v}{V_1}\right)dz,$$

$$\gamma_1 = \int_0^\infty \frac{v}{V_1}\left(1-\frac{u}{U}\right)dz, \quad \gamma_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{v}{V_1}\right)dz,$$

and $U = U_0 + xU_1$. These are exact, and the approximate forms for the Pohlhausen solution are obtained by replacing infinity in the integrals by a boundary-layer thickness δ.

The next step is to assume polynomial forms for the velocity distributions. If a quartic profile is used for the two-dimensional linearly retarded flow satisfying

$$u = 0, \quad \nu\frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{at} \quad z = 0;$$

$$u = U, \quad \frac{\partial u}{\partial z} = \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{at} \quad z = \delta;$$

it is known that separation is predicted at 0·156 although the exact value is 0·120. It appears that the reason for this poor agreement is due to the stringent condition $\partial^2 u/\partial z^2 = 0$ at the edge of the boundary layer. Indeed, it has been found by Jain & Bhatnagar (1962) that if instead we satisfy $\partial^3 u/\partial z^3 = 0$ at $z = 0$, separation is predicted at 0·112, which gives much better agreement.

Further, a calculation using the cubic profile which left $\partial^2 u/\partial z^2 = 0$ at $z = \delta$ unsatisfied indicated separation at about 0·13, which compares tolerably well with the exact value. This was considered accurate enough for the present investigation, and the following approximations were made:

$$\frac{u}{U} = \tfrac{1}{2}(3\gamma - \gamma^3) + \frac{\delta^2 U_1}{4\nu}(\gamma - 2\gamma^2 + \gamma^3),$$

$$\frac{v}{V_1} = \tfrac{1}{2}(3\gamma - \gamma^3) + \frac{\delta^2 V_1}{4\nu}(\gamma - 2\gamma^2 + \gamma^3) + F(X)(\gamma - 3\gamma^3 + 2\gamma^4),$$

where $\gamma = z/\delta$.

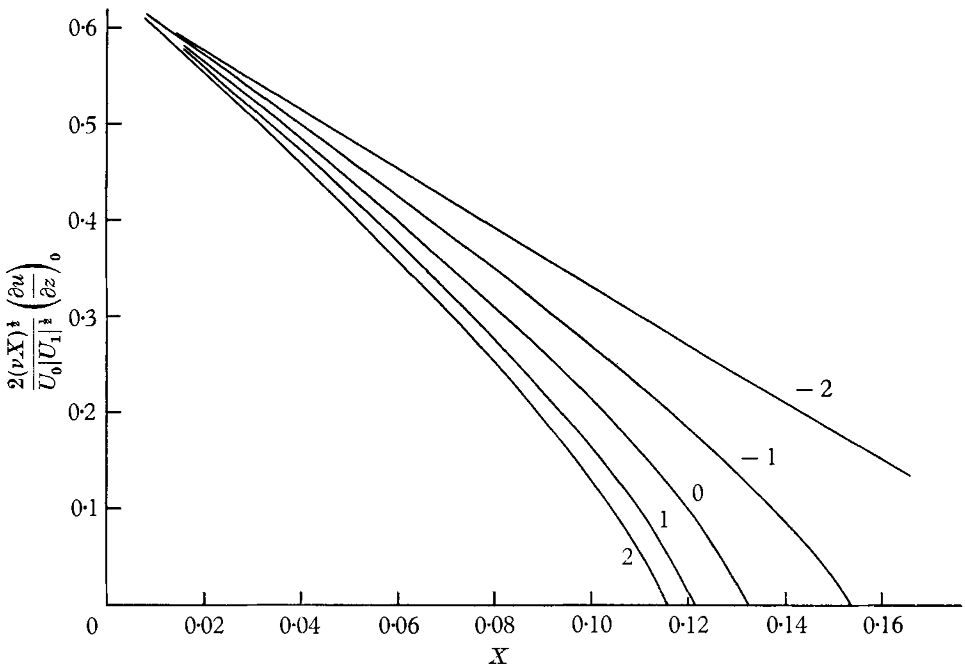

FIGURE 4. The x-component of skin friction for various values of α from the Pohlhausen results.

Substitution into the approximate form of (12) results in equations of the type

$$\frac{d\Delta}{dX} = f(X, \Delta, \alpha, F),$$

$$\frac{dF}{dX} = \frac{d\Delta}{dX} g_1(X, \Delta, \alpha, F) + g_2(X, \Delta, \alpha, F),$$

where $\Delta = \frac{1}{2}\delta(|U_1|/\nu X)^{\frac{1}{2}}$, for the determination of Δ and F. Similar equations are obtained when $U_1 = 0$, which corresponds to letting $\alpha \to \infty$.

For separation we are interested in the vanishing of $(\partial u/\partial z)_0$, i.e. in the possibility of $3 + 2BX\Delta^2$ vanishing. For the two-dimensional case of a linearly retarded external flow Δ remains finite and since $B = -1$ the boundary layer separates when $X = 3/2\Delta^2$. However, even with $V_1 < 0$ no separation appears possible with a linearly accelerated flow since $B = +1$. Indeed even when $U_1 = 0$ no boundary-layer separation occurs, as will be seen from the results presented in the following.

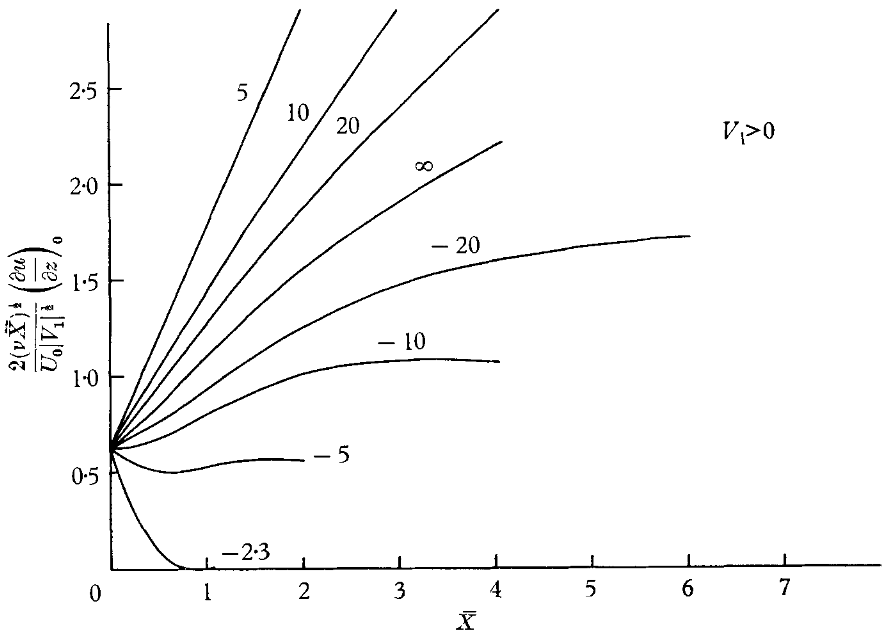

FIGURE 5. Pohlhausen results for $V_1 > 0$.

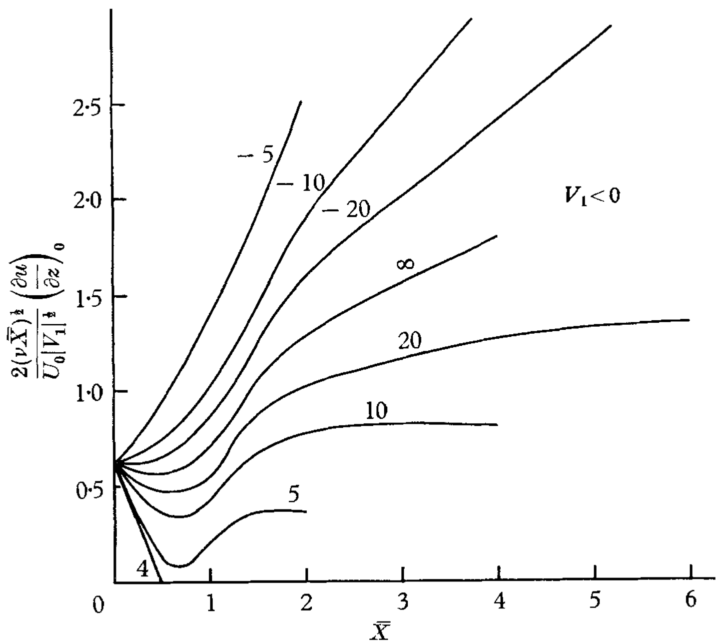

FIGURE 6. Pohlhausen results for $V_1 < 0$.

Pohlhausen results

As indicated previously both the one and two boundary-layer thickness methods gave virtually the same results and hence only one set is presented here.

Figure 4 gives a plot of $[2(\nu X)^{\frac{1}{2}}/U_0|U_1|^{\frac{1}{2}}]\,(\partial u/\partial z)_0$ against X for values of α in the range $-2 \leqslant \alpha \leqslant +2$. For $\alpha = 0$ separation is predicted at about $X_s = 0{\cdot}132$, whereas for $\alpha = +1, -1$ we obtain $X_s = 0{\cdot}121, 0{\cdot}154$ respectively. The actual

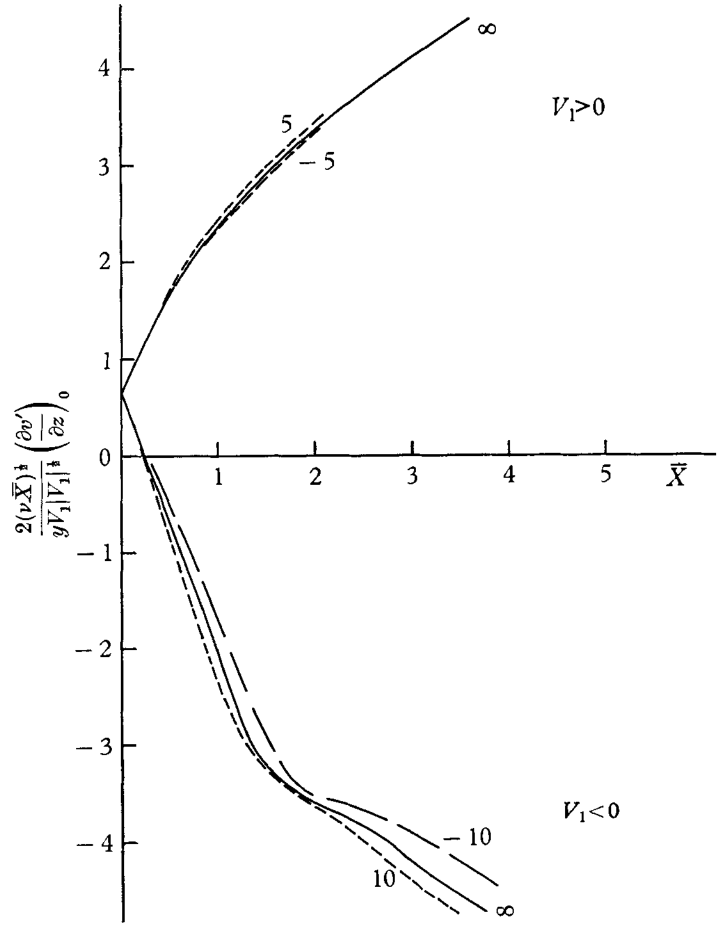

FIGURE 7. The variation of the transverse skin friction from the Pohlhausen results for $V_1 \gtrless 0$. The values of α are indicated on the figure.

movement of the position of separation for $|\alpha|$ small is found to be in reasonable agreement with Wilkinson's (1954) results, i.e. for $\alpha = -0{\cdot}1$ separation is delayed by about $0{\cdot}002$. However, for $\alpha \geqslant 4{\cdot}5$ and $\alpha \leqslant -2{\cdot}3$ the skin friction in the x-direction does not vanish, and hence it suggests that boundary-layer separation is completely inhibited.

Figures 5 and 6 contain plots of $[2(\nu\bar{X})^{\frac{1}{2}}/U_0|V_1|^{\frac{1}{2}}]\,(\partial u/\partial z)_0$ against $\bar{X}$ for the limiting cases $|\alpha| = \infty$, $V_1 > 0$ and $|\alpha| = \infty$, $V_1 < 0$ respectively. Also shown on these graphs are the results for $\alpha = -2{\cdot}3, \pm 5, \pm 10, \pm 20$ and $\alpha = 4, \pm 5, \pm 10,$

± 20 respectively. Figure 7 gives plots of $[2(\nu\bar{X})^{\frac{1}{2}}/yV_1|V_1|^{\frac{1}{2}}](\partial v'/\partial z)_0$ for a number of values of α.

It would appear that the plausible behaviour argued in §2 is confirmed.

5. Comparison and discussion

Pohlhausen and series solution

We are here concerned with the qualitative rather than the quantitative agreement, for, as mentioned above, the two-dimensional linearly retarded boundary-layer flow (i.e. $\alpha = 0$) is predicted to separate at $X_s = 0{\cdot}13$ by the Pohlhausen method, whereas the exact series solution gives $X_s = 0{\cdot}12$.

A useful guide concerning the qualitative agreement is obtained by comparing certain critical values of α. For example, that value of α, say α_e, corresponding to the earliest possible separation (i.e. gives $X_s(\alpha)$ its minimum value). The Pohlhausen method gives $\alpha_e = 2{\cdot}5$, whereas the series method (in conjunction with Howarth's modification) indicates a value near $1{\cdot}7$. There are two possible reasons for the poor agreement: the first is that the Pohlhausen method is just not good enough, particularly the nearer the leading edge that separation occurs. Secondly, it is possible that the series solution may not be very accurate for values of α near 2, since in this region (of α) the method used to find the skin friction and separation is breaking down owing to the non-existence of similarity in the functions concerned. However, it is more than likely that the series solution is more representative of the flow field than is the Pohlhausen solution.

Another possibility for comparison is that value of α, α_ℓ (where $\alpha_\ell < 0$), beyond which the boundary-layer separation is completely inhibited. The Pohlhausen results indicate $\alpha_\ell = -2{\cdot}3$ and, although the series method is unable to give a definite bound, we may note that Davey's (1961) results provide a check, and the agreement is very good. Further discussion is made in what follows.

Graphs are given in figures 8 and 9 showing values of $[2(\nu X)^{\frac{1}{2}}/U_0|U_1|^{\frac{1}{2}}](\partial u/\partial z)_0$ and $[2(\nu X)^{\frac{1}{2}}/yV_1|U_1|^{\frac{1}{2}}](\partial v'/\partial z)_0$ for various values of α. Further details are given in Banks (1963).

Although the agreement is fairly poor near separation, and particularly so in the case of $[2(\nu X)^{\frac{1}{2}}/V_1|U_1|^{\frac{1}{2}}](\partial v/\partial z)_0$, it is possible that for the non-separating flows the 'asymptotic' behaviour may be fairly accurate (see below).

Discussion with further comparison

It would appear that the only flows of real interest concern the boundary layer characterized by the linearly retarded flow in the x-direction (i.e. $U_1 < 0$), and so all that follows is solely concerned with the latter and also with the limiting flows governed by $U = U_0$, $V = yV_1$.

The boundary layers discussed here result from an external flow past a sharp leading edge and, with the exception of the limiting flows mentioned above, all the external flows considered have a stagnation point at $X = 1$ (i.e. $x = |U_1|/U_0$).

Now in the two-dimensional problem, and also in cases where $|\alpha|$ is not greater than about 2, it appears the boundary layer separates long before the stagnation point is reached. If we denote by X_s the value of X at which $(\partial u/\partial z)_0$ vanishes, then $(X_s, 0, 0)$ is a singular point of the differential equations which govern the surface streamlines, and the classification of such a point depends on the sign of $J = \partial(\epsilon_x, \epsilon_y)/\partial(x, y)$.

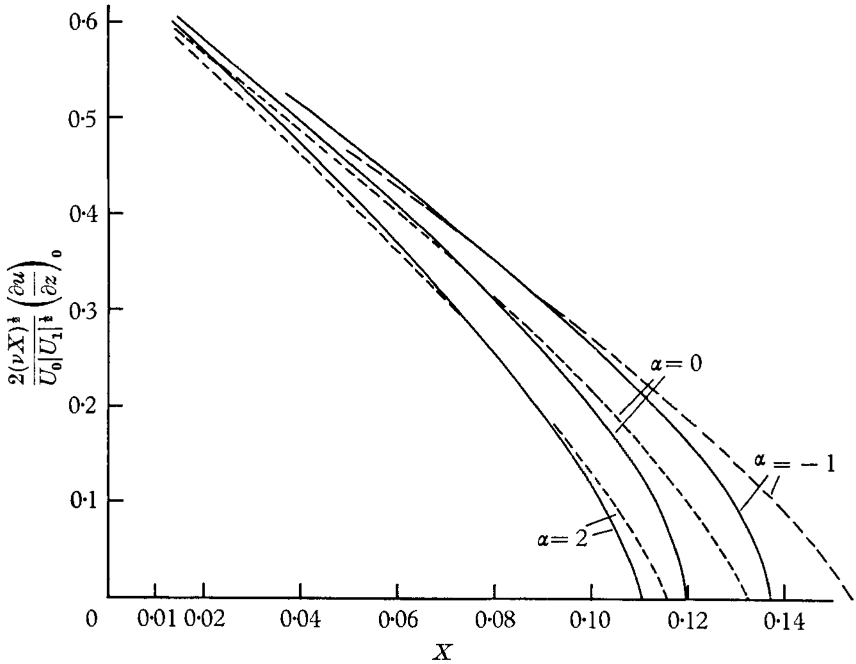

FIGURE 8. Comparison of x-component of skin friction. ——, series results; ------, Pohlhausen results.

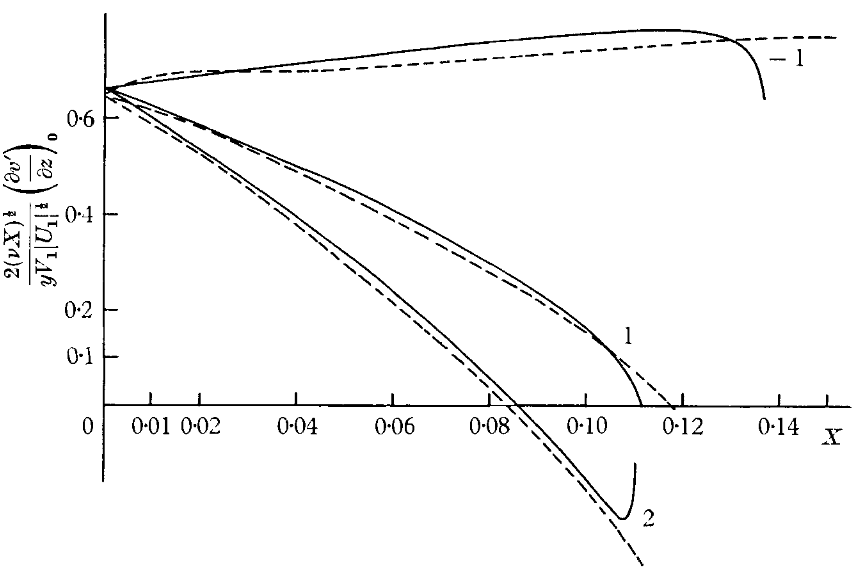

FIGURE 9. Comparison of transverse skin friction. ——, series results; ----, Pohlhausen results.

However, the curves in figure 3 appear to indicate that, for all relevant values of α, not only does $(\partial u/\partial z)_0$ vanish at X_s, but so also does $(\partial v/\partial z)_0$. Indeed, examination of the differential equation for M in (11) shows that the vanishing of $(\partial v/\partial z)_0$ is a direct consequence of the vanishing of $(\partial u/\partial z)_0$ and that in the neighbourhood of $X = X_s$ the solution of (11) for $\alpha = -1$ is of the form

$$L = 1{\cdot}85x^{\frac{1}{2}} + 5{\cdot}90x - \tfrac{4}{7}M_1 x^{\frac{5}{4}} + 52{\cdot}5x^{\frac{3}{2}} + \ldots$$

and

$$M = M_1 x^{\frac{1}{4}} - 109x^{\frac{1}{2}} + 29{\cdot}6M_1 x^{\frac{3}{4}} + (0{\cdot}206M_1^2 - 2037)\,x + 360M_1 x^{\frac{5}{4}} + \ldots,$$

where $x = X_s - X$ and M_1 is a disposable constant.

Near $x = 0$ the first two terms in the expansion of L are found to be sufficient to repeat the computer results reasonably well, although the convergence of M is such that each of the 5 terms indicated above give contributions of equal order of magnitude even at $x = 0{\cdot}0017$. If values of M are plotted for different α, it seems to suggest the singularity is $\frac{1}{4}$ rather than $\frac{1}{2}$ (i.e. $M_1 \neq 0$).

It would appear therefore that in the vicinity of $X = X_s$ either a lot more terms are required to obtain convergent results in the above series for M, or the step-lengths used in the integration should be reduced. However, since the singularity is a product of the method of solution and in particular of the assumption of similarity for the 'error' terms, its actual existence in the boundary-layer equations should be first confirmed by some other method (e.g. using finite differences). Of course one could also apply this argument to the x-component of skin friction, although the absence of such a singularity would be very surprising.

The fact that the singularity has been 'built-in' is also the case in Curle's (1960) two-dimensional investigation into the flows $U(x) = U_0(X - X^3)$ and $U(X) = U_0(X - X^3 + \beta X^5)$, where $\beta \doteqdot 0{\cdot}079$ and $-0{\cdot}122$.

A three-dimensional problem involving separation concerns the yawed cylinder and, although surprisingly no detailed examination has been attempted, a series investigation was made by Sears (1948) for $U = U_0(X - X^3)$, $V =$ constant. Four terms in the expansion for the span-wise velocity component were obtained and, since these suggest some form of similarity, it is conceivable that one could have proceeded with a Howarth-type investigation near separation. The above appears to indicate that such an investigation would lead to $(\partial v/\partial z)_0$ vanishing at $X = X_s$.†

However, for the present flows, it should be noted that since $(\partial^2 v/\partial z^2)_0 < 0$ it would appear that at X_s, where it is assumed $(\partial v/\partial z)_0 = 0$, the velocity profile of v (for certain values of α) is of the form shown in figure 10. Although there is no obvious reason why this is not a possibility it does mean that at some $X < X_s$ the stress and velocity both vanish in the interior of the fluid.

For the classification of the separating flows in this investigation, we assume that in the vicinity of separation

$$\frac{4}{U_0}\left(\frac{\nu X}{|U_1|}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial z}\right)_0 = L_1 x^{\frac{1}{2}}, \quad \frac{4}{V_1}\left(\frac{\nu X}{|U_1|}\right)^{\frac{1}{2}}\left(\frac{\partial v'}{\partial z}\right)_0 = yM_1 x^{\frac{1}{4}},$$

† A recent investigation by Mr P. G. Williams of N.P.L. into the flow $U = U_0(1 - X)$, $V =$ constant confirms that $(\partial v/\partial z)_0$ is singular at $X = X_s$ but that it remains positive.

where L_1, M_1 are both positive constants. Hence we find $J \to -\infty$ as $x \to 0$ so that the singular point $(X_s, 0, 0)$ is a saddle point. Further it can readily be shown that the normal velocity near $z = 0$ is positive so that the singular points are saddle points of separation.

The above discussion concerning the classification of singular points is based as usual on the topography of the surface streamline pattern, although it is found convenient also to classify in terms of the external flow velocities, since they will determine at least the leading terms in the behaviour of the normal velocity.

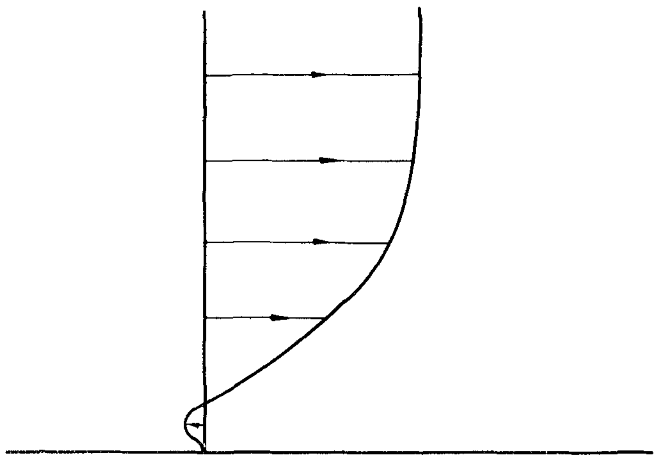

FIGURE 10

In the case of the separating boundary-layer flows the actual existence of the stagnation point in the external flow is of academic interest, since the singular point at separation must be the end-point of any boundary-layer calculation. However, when $|\alpha|$ is large and finite no separation appears possible and the stagnation point is then within the 'flow field'. Hence this stagnation point can also be regarded as a singular point of the appropriate differential equations and, with the x-component of the external velocity specified, its classification depends not only on the sign of V_1 but also on the relative magnitude of $|V_1|$ and $|U_1|$ of course.

For $V_1 < 0$ the point is a nodal point of separation, while for $V_1 > 0$ the point is a saddle point of attachment or separation according as $V_1 > -U_1$ or $V_1 < -U_1$. These cases are discussed in the following.

Saddle point of attachment ($V_1 > -U_1$)

As indicated earlier, saddle points of attachment have been discussed by Davey (1961) and it is of interest to compare his results with the limiting behaviour as $X \to 1$ of some of the flows discussed herein.

Davey postulated the external flow to be of the form $\bar{U} = \bar{a}\bar{x}$ and $\bar{V} = \bar{b}\bar{y}$, where $\bar{a} > 0 > \bar{b}$ and $\bar{b} \geq -\bar{a}$; the latter condition is necessary to ensure that the saddle-point flow is one of attachment. The barred variables have been introduced to avoid confusion in what follows since x and y in this work are to be respectively identified with $\bar{y}$ and $\bar{x}$ after a suitable change of origin.

Before proceeding to a detailed comparison it is of interest to note that Davey found that flow reversal in the $\bar{y}$-direction occurred for $\bar{c} = \bar{b}/\bar{a} \leqslant -0{\cdot}4294$, whereas the Pohlhausen results of §4 indicate that in the analogous régime separation occurs in the x-direction for $\alpha > a_\rho = -2{\cdot}3$. Hence considering the agreement between the comparable critical values of α and $1/\bar{c}$ ($-2{\cdot}3$ and $-2{\cdot}33$ respectively) we may anticipate similar agreement at other negative values of α for which the boundary layers do not separate.

				Skin-friction coefficients		Two-dimensional displacement thickness	
α	$\bar{c}$	X	$\bar{X}$	$\dfrac{\nu^{\frac{1}{2}}}{U_0 V_1^{\frac{1}{2}}}\left(\dfrac{\partial u}{\partial z}\right)_0$	$\left(\dfrac{\nu}{V_1^3}\right)^{\frac{1}{2}}\left(\dfrac{\partial v}{\partial z}\right)_0$	$\left(\dfrac{V_1}{\nu}\right)^{\frac{1}{2}}\delta_x$	$\left(\dfrac{V_1}{\nu}\right)^{\frac{1}{2}}\delta_y$
$-\infty$	0	—	2·0	0·553	1·195	1·018	0·645
			3·0	0·553	1·267	1·018	0·641
			4·0	0·553	1·267	1·018	0·641
				(0·570)	(1·233)	(1·026)	(0·648)
-10	$-0{\cdot}1$	0·2	2·0	0·447	1·186	1·134	0·651
		0·3	3·0	0·447	1·186	1·134	0·651
		0·4	4·0	0·447	1·186	1·134	0·651
				(0·459)	(1·228)	(1·141)	(0·654)
-5	$-0{\cdot}2$	0·4	2·0	0·330	1·178	1·284	0·658
		0·6	3·0	0·330	1·178	1·283	0·659
				(0·335)	(1·226)	(1·287)	(0·658)
-4	$-0{\cdot}25$	0·4	1·6	0·267	1·175	1·375	0·661
				(0·268)	(1·225)	(1·375)	(0·659)
$-2{\cdot}5$	$-0{\cdot}4$	0·4	1·0	0·060	1·170	1·727	0·663
				(0·046)	(1·226)	(1·702)	(0·663)

TABLE 3

It will be appreciated that in order to compare with Davey's results it is important that the values of X at which this is to be made are not small, and, since we can hardly expect the series results for these non-separating flows to be convergent away from the vicinity of the leading edge, we shall concern ourselves with a comparison of the Pohlhausen results. Also since the latter are only approximate this comparison is made at $X = \frac{1}{2}$. This only refers to α finite, although even for the limiting case when $\alpha = -\infty$ we assume that 'asymptotic' behaviour prevails at $\bar{X} = O(1)$.

Davey gives the skin friction and the two-dimensional displacement thickness associated with both components of the velocity for the values $\alpha = 1/\bar{c} = -2{\cdot}5$, -4, -5, -10, and the limiting case as $\alpha \to -\infty$ is given in Howarth's original paper on the subject. The two-dimensional displacement thickness for the x-component of velocity is defined as

$$\delta_x = \int_0^\infty \left(1 - \frac{u}{U}\right) dz$$

and a similar expression for the y-component.

Although the case for $\alpha = -\infty$ (i.e. $\bar{c} = 0$) requires separate treatment, the details are straightforward; the results are given in table 3, in which Howarth's and Davey's 'asymptotic' values are in brackets.

The agreement here is so good that Davey's method of treating the problem of saddle-point flows as locally determinate is completely justified. This, of course, only refers to the non-separating flows.

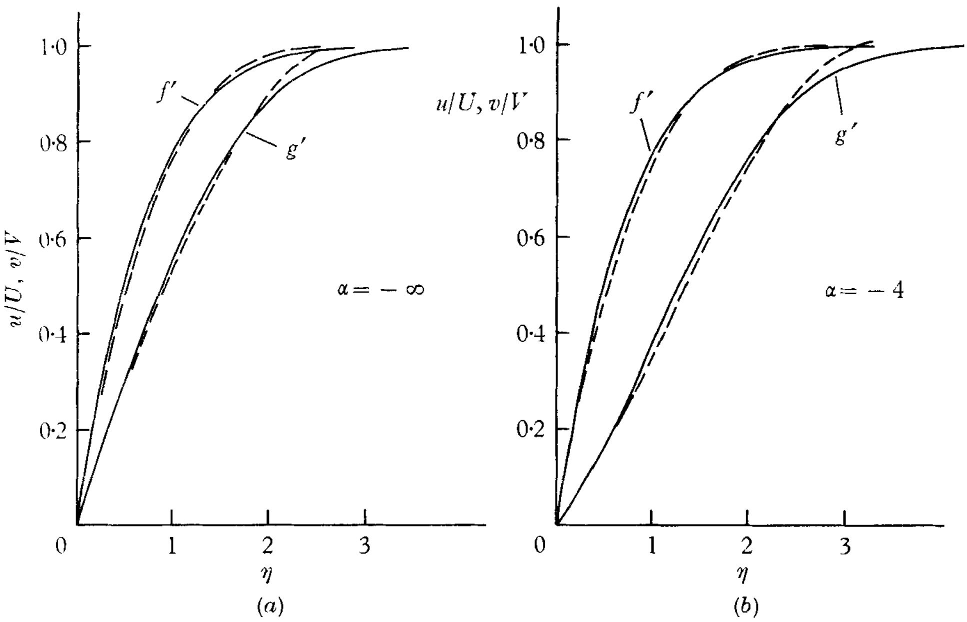

FIGURE 11. Comparison of velocity profiles for values of α shown. ——, exact results; ----, Pohlhausen results. (a) Shows Howarth's profiles and (b) shows Davey's profiles.

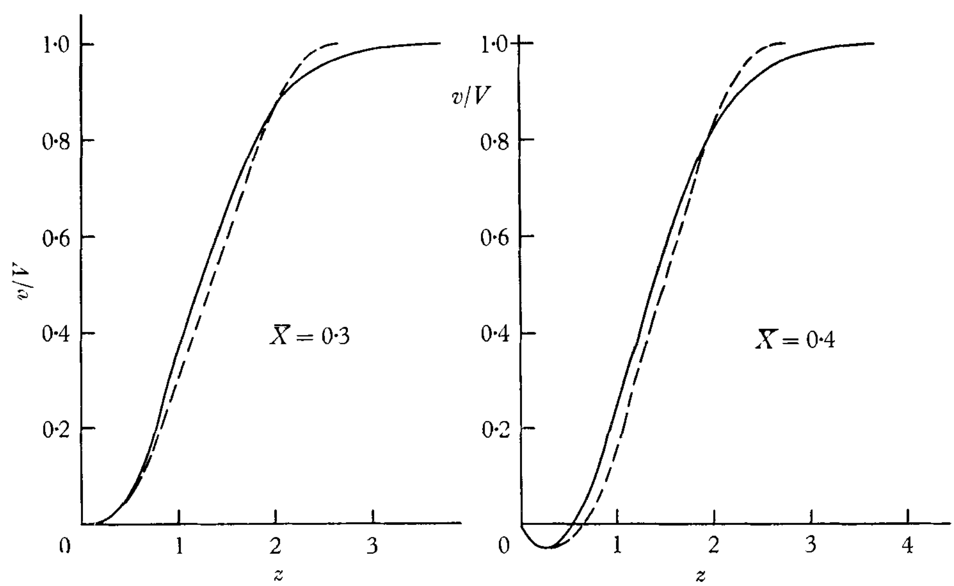

FIGURE 12. Transverse velocity profiles for $|\alpha| = \infty$ for $\bar{X} = 0{\cdot}3$ and $0{\cdot}4$. ——, series; ----, Pohlhausen.

Howarth (1951) and Davey (1961) also give the velocity profiles for $\alpha = -\infty$ and $\alpha = -4$ respectively and a comparison is given in figure 11, where the independent variable $\eta = (\bar{u}/\nu)^{\frac{1}{2}}z = 2(|\alpha|X)^{\frac{1}{2}}z$ is used. Agreement is again found to be very reasonable.

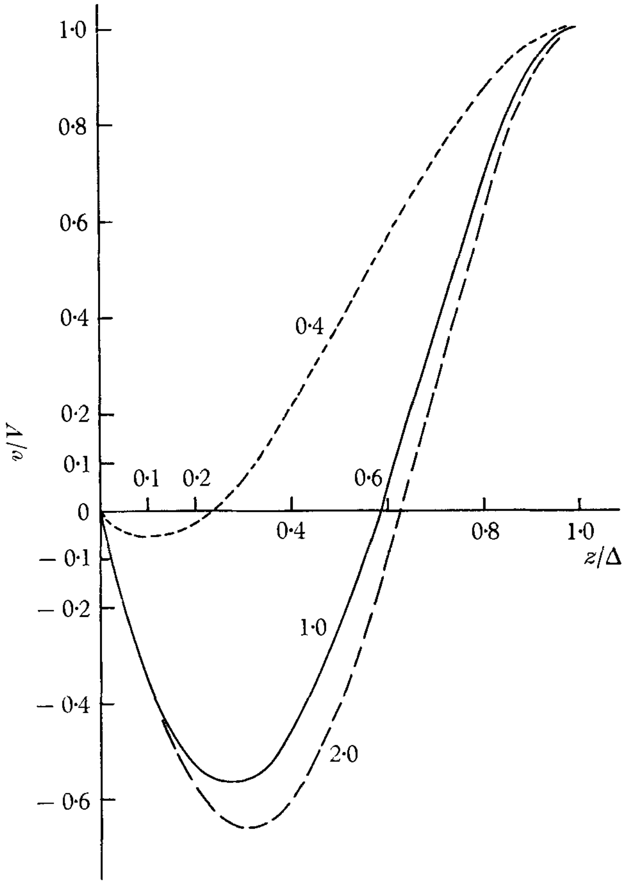

FIGURE 13. Transverse velocity profiles at $\bar{X} = 0{\cdot}4$, 1·0 and 2·0 from Pohlhausen results.

Nodal points of separation ($V_1 < 0$)

This section is concerned with the boundary layers governed by the external velocities $U = U_0 + xU_1$ and $V = yV_1$ with U_1, V_1 both < 0, and is further restricted to flows for which α is greater than about 3, since for other values of $\alpha\,(\geqslant 0)$ it appears the boundary layer separates.

An important feature of the non-separating and some of the separating flows concerns the path of a fluid particle near $z = 0$, or, equivalently, the surface streamlines. It seems that, although near the leading edge (i.e. X small) the boundary-layer flow is uni-directional, it is found that away from this edge the flow becomes bi-directional for α greater than a certain value, α_c say ($\alpha_c \doteqdot 1$). Furthermore, in the case of flows that separate this flow reversal in the boundary layer occurs only in the immediate vicinity of separation.

The y-component of the boundary-layer velocity profile corresponding to the external flow velocities $U = U_0$, $V = -y|V_1|$ is given in figure 12 together with the comparable Pohlhausen results. These are plotted at $\bar{X} = 0{\cdot}3$ and $0{\cdot}4$ as indicated; other Pohlhausen profiles are given for $\bar{X} = 1{\cdot}0$, and $2{\cdot}0$ in figure 13. It will be apparent that, near the 'wall' $z = 0$, the y-component of velocity changes sign for $\bar{X}$ large enough, so that the flow in the (y, z)-plane is of the form shown in figure 14. This is also the type of flow occurring when $U = U_0 - x|U_1|$ and $V = -y|V_1|$, providing $\alpha > \alpha_c$, and appears to suggest that even for stagnation point flow of separation the local flow near $y = z = 0$ is similar to the case when $V = y|V_1|$ corresponding to stagnation point flow of attachment.

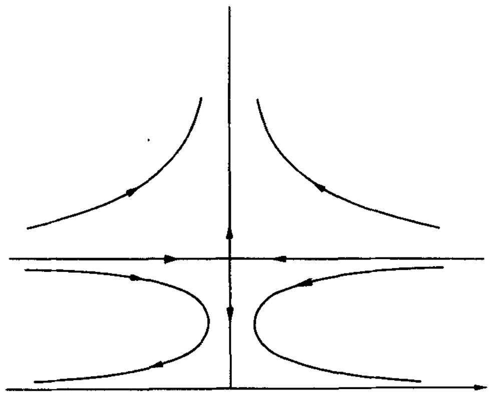

FIGURE 14

In actual fact this is precisely the same sort of flow as was found by Proudman & Johnson (1962) in their investigation dealing with the two-dimensional boundary-layer growth at a stagnation point of separation. As they point out there is no steady solution of this boundary-layer problem, and they have made theirs determinate by considering it as an impulsive boundary-layer growth investigation, being locally determinate in the sense outlined above.

In the present work the problem has been made soluble by a transverse field in the x-direction. Indeed, there appears to be an analogy between the two problems if we identify the time variable of Proudman & Johnson (1962) with the space variable x used herein.

Even in the case of non-separating boundary-layer flows with $|\alpha|$ finite the results of this section will not be valid right up to the stagnation point of course. This is not surprising since there is no steady solution for flow at such a three-dimensional stagnation point of separation.

These results form part of a thesis submitted to the University of Bristol.

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6. Addendum

Some time after this paper had been submitted a detailed comparison of the results here with those of Sowerby (1965) was made and it was noticed that the boundary-layer calculation for the case $\alpha = 1$ could be reduced to a two-dimensional one.

It appears that for $V_1 = U_1$ (i.e. $\alpha = 1$) a solution of the y-component momentum equation in (2) is given by

$$v = \frac{U_1 u}{U_0 + xU_1},$$

so that the equations which determine u and w are

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = U(x)\frac{dU(x)}{dx} + \nu\frac{\partial^2 u}{\partial z^2},$$

$$\frac{\partial u}{\partial x} + \frac{U_1 u}{U(x)} + \frac{\partial w}{\partial z} = 0,$$

subject to the boundary conditions

$$u = w = 0 \quad \text{on} \quad z = 0,$$

$$u \to U(x) = U_0 + xU_1 \quad \text{as} \quad z \to \infty.$$

Reduction to two-dimensionality is then achieved, of course, by an application of Mangler's transformation.

As far as the universal functions are concerned, the existence of this special solution does mean that more computing was done than was actually necessary, although on the other hand it does provide certain checks on the results presented here. It also means that it is possible to use the results of two-dimensional boundary-layer theory as a comparison; for example, if $U_1 < 0$ an application of the method proposed by Thwaites (1949) predicts separation at $X_s = 0{\cdot}1126$, whereas the present work gives $X_s = 0{\cdot}1113$.

For this special solution it follows that the streamlines form a system of planes which are normal to the plane $z = 0$ and which intersect the latter in the lines $(U_0 + xU_1)/y =$ constant. These 'streamsheets' radiate from or to $X = \mp 1$ depending on whether $U_1 > 0$ or < 0. Also for $U_1 < 0$ $(\partial v/\partial z)_0$ vanishes like $(\partial u/\partial z)_0$.

Although the radiating streamline pattern suggests cylindrical polar co-ordinates, the boundary conditions, and the solution, are most easily described in Cartesians.

If we interpret this boundary layer as formed by a uniform stream over a suitably shaped body, a theorem due to L. C. Squire (Rosenhead 1963, p. 457) then implies that the curves $(U_0 + xU_1)/y =$ constant are geodesics of that surface. The axisymmetric forms of the reduced equations are not unexpected, for it is a well-known result that, in writing the equations in terms of co-ordinates which are based on the projection of the external streamlines, the resulting equations reduce to this form if there is no secondary flow.

Finally, the pertinent question presents itself: are there other mainstreams (U, V) for which $v'/V = u'/U$? An answer to this question for the boundary layer on a developable surface is readily obtained using equations (1) and it appears that the condition to be satisfied is that

$$\left(\frac{U}{V}\right)\frac{\partial}{\partial x}\left(\frac{U}{V}\right)+\frac{\partial}{\partial y}\left(\frac{U}{V}\right) = 0.$$

The general solution of this equation is

$$Vx = yU + VG(U/V),$$

where G is some arbitrary function. The particular function $G \equiv -U_0/U_1$ yields

$$\frac{U}{V} = \frac{U_0 + xU_1}{yU_1}.$$

Apart from the other (trivial) case $G \equiv 0$, which gives rise to the axisymmetric stagnation point flow, no other interesting examples have been found.

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